n-LINKED FIELDS OF CHARACTERISTIC 2

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1. Linkage and other properties

To avoid certain problems, we assume throughout the talk that F is nonreal (which goes without saying when $\text{char}(F) = 2$) and that F admits division quaternion algebras.

In this case, $u(F)$ is the maximal dimension of an anisotropic nonsingular quadratic form over F. By the assumption, $u(F) \geq 4$.

We say that a field F is n-linked when every n quaternion algebras over F share a quadratic splitting field. A 2-linked field is thus what is known in the literature as a "linked field".

This notion proved to have connections to other field properties and invariants:

- When F is linked, its u-invariant is either 4 or 8, and it is 4 if and only if $I_q^3 F = 0$. (Elman & Lam in 1972 for $\text{char}(F) \neq 2$ and Chapman & Dolphin in 2017 for $\text{char}(F) = 2$, based on some results from Faivre's PhD thesis from 2006.)
- $u(F) = 4$ if and only if F is 3-linked. (Becher, Chapman, Dolphin & Leep in 2018.)

Example 1.1.

- $\mathbb{C}(x, y)$ *is 3-linked. (Show argument using* C_2*-property)*
- $\mathbb{C}(\!(x)\!)(\!(y)\!)(\!(z)\!)$ *is 2-linked but not 3-linked.*
- $\mathbb{C}(x, y, z)$ *is not 2-linked.*
- $\mathbb{Q}[i]$ *is n-linked for any* $n \in \mathbb{N}$.
- $\mathbb{F}_p(x)$ *is n-linked for any* $n \in \mathbb{N}$
- $\mathbb{F}_p^{sep}(x, y)$ *is 3-linked.*

Question 1.2. *Does 3-linkage imply n-linkage for any* $n \in \mathbb{N}$*?
And in particular, are* $\mathbb{C}(x, y)$ *and* $\mathbb{F}_p(x, y)$ *4-linked?*

2. $\mathbb{C}(x, y)$

The proof that this field is not 4-linked was obtained jointly with Tignol (2019).

Consider the quaternion algebras $Q_1 = (x, y)$, $Q_2 = (x, y + 1)$, $Q_3 = (x + 1, y)$ and $Q_4 = (x, xy + 1)$, and the pure parts of their norm forms $\varphi_1, \varphi_2, \varphi_3, \varphi_4$. If the algebras share a quadratic splitting field, then there exists a solution to the system

$\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4$. However, then there exists a solution to the underlying system over the residue field $F = k(x, y)$ with respect to the extension of the dyadic valuation from $\mathbb{Q}$ to $\mathbb{C}(x, y)$. But the values the forms represent form F^2-vectors spaces $V_1 = \mathrm{Sp}\{x, y, xy\}$, $V_2 = \mathrm{Sp}\{x, y+1, xy\}$, $V_3 = \mathrm{Sp}\{x+1, y, xy\}$ and $V_4 = \mathrm{Sp}\{x, y, xy+1\}$, and so these spaces must have a nontrivial intersection, and they don't.

3. $\mathbb{F}_2^{sep}(x, y)$

When one considers the algebras $Q_1 = [x^{-1}, y)$, $Q_2 = [y^{-1}, x)$ and $Q_3 = [x^{-1}y^{-1}, y)$, it is immediate from the intersection of the values of the trace zero elements that the algebras do not share an inseparable field extension of $F = \mathbb{F}_2^{sep}(x, y)$. However, they do share a separable extension. The question is which algebra can we add to obtain a quadruple without a common quadratic splitting field? And more importantly, how do we prove that?

The tool we need turns out to be the w-invariant defined by Tignol: $w(D) = \min\{\mathfrak{v}(\mathrm{Tr}(z)) - \mathfrak{v}(z) : z \in D\}$. In particular, $w(Q_1) = (\frac{1}{2}, 0)$, $w(Q_2) = (0, \frac{1}{2})$ and $w(Q_3) = (\frac{1}{2}, \frac{1}{2})$. The separable subspaces of Q_1 have w-invriant either $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2}) \cdot t$ or $(\frac{1}{2}, \frac{1}{2}) \cdot t$ for some $t \in \mathbb{N}$. The separable subspaces of Q_2 have w-invariant either $(0, \frac{1}{2})$ or $(\frac{1}{2}, 0) \cdot (t+1)$ or $(\frac{1}{2}, \frac{1}{2}) \cdot t$ for some $t \in \mathbb{N}$. The separable subspaces of Q_3 have w-invariant either $(\frac{1}{2}, \frac{1}{2})$ or $(\frac{1}{2}, 0) \cdot (t+1)$ or $(0, \frac{1}{2}) \cdot (t+1)$ for some $t \in \mathbb{N}$. Therefore, every common separable subspace of Q_1, Q_2 and Q_3 must have w-invariant $(\frac{1}{2}, \frac{1}{2})$. Therefore, for any Q_4 with $w(Q_4) > (\frac{1}{2}, \frac{1}{2})$, the four algebras do not share any maximal subfield. Such Q_4 exists, for instance $Q_4 = [x^{-2}y^{-1}, y)$.

4. Open questions

- Is $\mathbb{F}_p^{sep}(x, y)$ not 4-linked for odd p?
- Are there fields which are 4-linked and not 5-linked?
- Over $\mathbb{C}(x, y)$, every two cyclic algebras of degree 3 share a common subfield. Does that extend to higher degrees?
- In a more recent preprint, I showed that for any odd prime p, there exist p^2 cyclic algebras over $\mathbb{C}(x, y)$ with no common maximal subfield. Can this number be reduced?